Modeling Polynomial Functions of Two Discrete Variables

Warren P. Adams* and Stephen M. Henry**

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** System Readiness and Sustainment Technologies Group, Sandia Natl. Labs.

July 24, 2014
"So the last will be first"

Matthew 20:16.
"So the last will be first"
"and the first will be last."

Matthew 20:16.
Outline

1. Overview
   - Problem of interest
   - Classical approaches
   - Objective

2. Focus on a single variable
   - Use of LIPs
   - Examples of LIPs
   - Property 1 of LIPs

3. Focus on two variables
   - Property 2 of LIPs
   - Property 3 of LIPs
   - Use of projections

4. Question and final comments
Quadratic expressions

The linearization of products of variables within optimization problems has been an active area of study for over 50 years. Given variables $x_1$ and $x_2$, how do we model the product $x_1 x_2$ when

- $x_1$ and $x_2$ are binary?
- $x_1$ and $x_2$ are continuous (bounded)?
- $x_1$ and $x_2$ are general discrete?
- $x_1$ and $x_2$ are a mix of the above three?
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A standard approach is to introduce a new variable $w_{12}$ which represents this product, and then form a polytope that enforces $x_1 x_2 = w_{12}$ at all extreme points.
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A standard approach is to introduce a new variable $w_{12}$ which represents this product, and then form a polytope that enforces $x_1 x_2 = w_{12}$ at all extreme points.
Fortet/Glover approach for binary products

Fortet/Glover inequalities

\[ w_{12} \geq 0 \]
\[ w_{12} \geq x_1 + x_2 - 1 \]
\[ w_{12} \leq x_1 \]
\[ w_{12} \leq x_2 \]

These inequalities give

\[ \text{conv}\{x_1 \in \{0, 1\}, x_2 \in \{0, 1\}, w_{12} = x_1 x_2\}. \]

Every binary product is associated with an extreme point, and the polytope enforces \( w_{12} = x_1 x_2 \) when \( x_1, x_2 \in \{0, 1\} \).
McCormick approach for continuous/discrete products

McCormick Inequalities

\[
\begin{align*}
    w_{12} & \geq l_2 x_1 + l_1 x_2 - l_1 l_2 \\
    w_{12} & \geq u_2 x_1 + u_1 x_2 - u_1 u_2 \\
    w_{12} & \leq u_2 x_1 + l_1 x_2 - l_1 u_2 \\
    w_{12} & \leq l_2 x_1 + u_1 x_2 - u_1 l_2
\end{align*}
\]

These inequalities give

\[
\text{conv}\{x_1 \in [l_1, u_1], x_2 \in [l_2, u_2], w_{12} = x_1 x_2\}.
\]

Every integer product is not associated with an extreme point, and the polytope does not enforce \(w_{12} = x_1 x_2\) when \(x_1, x_2 \in \{0, 1, 2\}\).
An optimization problem

How to solve the below as a linear program?

minimize \( \{(x_1 - 1)^2 + (x_2 - 1)^2 : x_1 \in \{0, 1, 2\}, \ x_2 \in \{0, 1, 2, 3\}\} \)

Desire a polytope having 12 extreme points to correspond to the 12 realizations of \((x_1, x_2)\), and all linearized variables to equal to their intended products at the extreme points.
A new family of polytopes

Let $x_1 \in S_1 \equiv \{\theta_{11}, \ldots, \theta_{1k_1}\}$ and $x_2 \in S_2 \equiv \{\theta_{21}, \ldots, \theta_{2k_2}\}$, assume in increasing order.

We wish to linearize polynomial expressions of the form

$$p(x_1, x_2) \equiv \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} a_{ij} x_1^i x_2^j.$$
A new family of polytopes

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We wish to linearize polynomial expressions of the form

$$p(x_1, x_2) \equiv \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} a_{ij} x_1^i x_2^j.$$ 

To do this, we introduce a family of discrete polytopes

$$DP(d_1, d_2) = \text{conv} \left\{ x_1 \in S_1, \; x_2 \in S_2, \begin{array}{l} w_{ij} = x_1^i x_2^j \quad \forall (i, j) \ni i + j \geq 2, \quad \forall i \ni \{0, \ldots, d_1\}, \; \forall j \ni \{0, \ldots, d_2\} \end{array} \right\}.$$
How to define such polytopes?
### Discrete polytopes - summary of results, round 1

**Objective**

\[ DP(d_1, d_2) \text{ for } x_1 \in \{\theta_{11}, \ldots, \theta_{1k_1}\}, \ x_2 \in \{\theta_{21}, \ldots, \theta_{2k_2}\}. \]

<table>
<thead>
<tr>
<th>( d_2 )</th>
<th>0</th>
<th>1</th>
<th>( 2 \leq d_2 \leq k_2 - 2 )</th>
<th>( d_2 = k_2 - 1 )</th>
</tr>
</thead>
<tbody>
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\[
p(x_1, x_2) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} a_{ij} x_1^i x_2^j
\]

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The Ohio State University, 2014 MIP Workshop
**Overview**
Focus on a single variable
Focus on two variables
Question and final comments

**Problem of interest**
Classical approaches
Objective

---

**Discrete polytopes - summary of results, round 1**

\[ DP(d_1, d_2) \text{ for } x_1 \in \{\theta_{11}, \ldots, \theta_{1k_1}\}, \; x_2 \in \{\theta_{21}, \ldots, \theta_{2k_2}\}. \]

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\[
p(x_1, x_2) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} a_{i,j} x_1^i x_2^j
\]

---

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Discrete polytopes - summary of results, round 1

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\[ p(x_1, x_2) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} a_{ij} x_1^i x_2^j \]

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\[
\begin{array}{|c|c|c|c|}
\hline
 & d_2 = 0 & d_2 = 1 & 2 \leq d_2 \leq k_2 - 2 & d_2 = k_2 - 1 \\
\hline
\hline
\text{ } & \text{constant} & \theta_{21} \leq x_2 \leq \theta_{2k_2} & & \\
\hline
\hline
\text{ } & \theta_{11} \leq x_1 \leq \theta_{1k_1} & \text{Glover/McCormick} & & \\
\hline
\hline
\text{ } & \theta_{11} \leq x_1 \leq \theta_{1k_1} & & & \\
\hline
\hline
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\hline
\end{array}
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| \(d_1 = k_1 - 1\) | ???? |

\[ p(x_1, x_2) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} a_{ij} x_1^i x_2^j \]
LIPs

Suppose that \( x \in \{\theta_1, \theta_2, \ldots, \theta_k\} \). Then there exist \( k \) Lagrange interpolating polynomials (LIPs), with each being a polynomial of degree \( k - 1 \).

\[
L_i(x) = \frac{\prod_{j=1, j \neq i}^{k} (x - \theta_j)}{\prod_{j=1, j \neq i}^{k} (\theta_i - \theta_j)} \quad \text{for all } i = 1, \ldots, k
\]

\[
xL_i(x) = \theta_i L_i(x) \quad \text{for all } i = 1, \ldots, k
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\]

[Adams, Sherali 2005] shows how LIPs can be used to construct a hierarchy of relaxations for mixed-discrete programs, culminating with an explicit convex hull description.
LIPs for a binary variable

Suppose $x \in \{0, 1\}$. Then there exist 2 LIPs, with each being a polynomial of degree 1.

\[
L_1(x) = 1 - x, \quad L_2(x) = x
\]
Example for a discrete variable

LIPs for a discrete variable

Suppose that a discrete variable $x$ can realize values in the set $S = \{-2, -1, 0, 1, 2\}$. Then there exist five LIPs of the form:

\[
L_1(x) = \frac{(x+1)(x-0)(x-1)(x-2)}{(-2+1)(-2-0)(-2-1)(-2-2)}
\]

\[
L_2(x) = \frac{(x+2)(x-0)(x-1)(x-2)}{(-1+2)(-1-0)(-1-1)(-1-2)}
\]

\[
L_3(x) = \frac{(x+2)(x+1)(x-1)(x-2)}{(0+2)(0+1)(0-1)(0-2)}
\]

\[
L_4(x) = \frac{(x+2)(x+1)(x-0)(x-2)}{(1+2)(1+1)(1-0)(1-2)}
\]

\[
L_5(x) = \frac{(x+2)(x+1)(x-0)(x-1)}{(2+2)(2+1)(2-0)(2-1)}
\]
Nonnegative LIPs in matrix form

Given $x \in \{-2, -1, 0, 1, 2\}$ the five LIPs, with nonnegativity enforced, are:

$$
\begin{bmatrix}
0 & \frac{1}{12} & -\frac{1}{2} & -\frac{1}{12} & \frac{1}{24} \\
0 & -\frac{2}{3} & -\frac{3}{2} & \frac{1}{6} & -\frac{1}{6} \\
1 & 0 & -\frac{5}{3} & 0 & \frac{1}{4} \\
0 & \frac{2}{3} & -\frac{4}{2} & 0 & -\frac{1}{6} \\
0 & -\frac{1}{12} & -\frac{1}{24} & \frac{1}{12} & \frac{1}{24}
\end{bmatrix}
\begin{pmatrix}
x \\
x^2 \\
x^3 \\
x^4
\end{pmatrix}
\geq
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
$$
Nonnegative, linearized LIPs

Given $x \in \{-2, -1, 0, 1, 2\}$, the five nonnegative, linearized LIPs are:

$$
\begin{bmatrix}
0 & \frac{1}{12} & -\frac{1}{24} & -\frac{1}{12} & -\frac{1}{24} \\
0 & -\frac{2}{3} & -\frac{1}{3} & 0 & -\frac{1}{6} \\
1 & 0 & -\frac{4}{3} & 0 & -\frac{2}{3} \\
0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{6} & -\frac{4}{3} \\
0 & -\frac{1}{12} & -\frac{1}{24} & \frac{1}{12} & \frac{1}{24}
\end{bmatrix}
\begin{bmatrix}
x \\
x^2 \\
x^3 \\
x^4
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0 & \frac{1}{12} & -\frac{1}{24} & -\frac{1}{12} & \frac{1}{24} \\
0 & -\frac{1}{3} & \frac{2}{6} & 0 & -\frac{1}{6} \\
1 & 0 & -\frac{2}{4} & 0 & \frac{1}{4} \\
0 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{6} & -\frac{1}{6} \\
0 & -\frac{1}{12} & -\frac{1}{24} & \frac{1}{12} & \frac{1}{24}
\end{bmatrix}
\begin{bmatrix}
x \\
x^2 \\
x^3 \\
x^4
\end{bmatrix}
\geq
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
$$

This polytope has the 5 extreme points given by:

$$
\{ x \}_{L} \in \left\{ 
\begin{bmatrix}
-2 \\
4 \\
-8 \\
16
\end{bmatrix},
\begin{bmatrix}
-1 \\
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
1 \\
1 \\
1 \\
4
\end{bmatrix},
\begin{bmatrix}
2 \\
8 \\
16
\end{bmatrix}
\right\}.
$$
Useful LIP property 1

Given $x_1 \in \{\theta_{11}, \ldots, \theta_{1k_1}\}$, suppose that we form all $k_1$ LIPs, set these polynomials nonnegative, and linearize the $k_1 - 2$ nonlinear terms to obtain a system of $k_1$ equations in $k_1 - 1$ variables.
Useful LIP property 1

Given $x_1 \in \{\theta_{11}, \ldots, \theta_{1k_1}\}$, suppose that we form all $k_1$ LIPs, set these polynomials nonnegative, and linearize the $k_1 - 2$ nonlinear terms to obtain a system of $k_1$ equations in $k_1 - 1$ variables.

The resulting system describes a polytope having $k_1$ extreme points, and each extreme point corresponds to some $\theta_{1i}$, with $w_{i0} = x_1^i$ for all $i = 1, \ldots, k_1$. [Adams, Sherali 2005]
Discrete polytopes - summary of results, round 2

\[ DP(d_1, d_2) \text{ for } x_1 \in \{\theta_{11}, \ldots, \theta_{1k_1}\}, \ x_2 \in \{\theta_{21}, \ldots, \theta_{2k_2}\}. \]

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\[
p(x_1, x_2) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} a_{ij} x_1^i x_2^j\]

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The task of finding $DP(d_1, 0)$ for $d_1 \in \{2, \ldots, k_1 - 2\}$ (equivalently $DP(0, d_2)$ for $d_2 \in \{2, \ldots, k_2 - 2\}$) amounts to characterizing projections of the polytope defined by the LIPs.
Projecting LIPs of a single variable

The task of finding $DP(d_1, 0)$ for $d_1 \in \{2, \ldots, k_1 - 2\}$ (equivalently $DP(0, d_2)$ for $d_2 \in \{2, \ldots, k_2 - 2\}$) amounts to characterizing projections of the polytope defined by the LIPs.

We introduce the notion of type-0 and type-1 bit strings. For a $k_1$-length bit string with $d_1$ elements of value 1, we have a

- type-0 string if every 0 has an **even** number of 1’s to its right.
- type-1 string if every 0 has an **odd** number of 1’s to its right.
Projecting LIPs of a single variable

The task of finding $DP(d_1, 0)$ for $d_1 \in \{2, \ldots, k_1 - 2\}$ (equivalently $DP(0, d_2)$ for $d_2 \in \{2, \ldots, k_2 - 2\}$) amounts to characterizing projections of the polytope defined by the LIPs.

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- type-0 string if every 0 has an even number of 1’s to its right.
- type-1 string if every 0 has an odd number of 1’s to its right.

What do these strings have to do with projecting LIPs???
Recall the LIPs for \( x_1 \in \{-2, -1, 0, 1, 2\} \) where \( k_1 = 5 \). These LIPs give the 4-dimensional polytope \( DP(4, 0) \).

\[
\begin{bmatrix}
0 & \frac{1}{12} & -\frac{1}{24} & -\frac{1}{12} & -\frac{1}{24} \\
0 & -\frac{2}{3} & \frac{2}{3} & \frac{1}{6} & -\frac{1}{6} \\
1 & 0 & -\frac{5}{4} & 0 & \frac{1}{4} \\
0 & \frac{2}{3} & -\frac{3}{2} & -\frac{1}{6} & -\frac{1}{6} \\
0 & -\frac{1}{12} & -\frac{1}{24} & \frac{1}{12} & \frac{1}{24}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_1^2 \\
x_1^3 \\
x_1^4
\end{bmatrix}
\geq
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

What if we want to project down onto the first 3 dimensions? This would give \( DP(3, 0) \).
We have $k_1 = 5$ and $d_1 = 3$, so an example of a type-0 and type-1 string is given by

\[
\text{type-0: } 1 \ 0 \ 1 \ 1 \ 0 \quad \text{type-1: } 0 \ 1 \ 1 \ 0 \ 1
\]
We have \( k_1 = 5 \) and \( d_1 = 3 \), so an example of a type-0 and type-1 string is given by

\[
\begin{align*}
\text{type-0:} & \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \\
\text{type-1:} & \quad 0 \quad 1 \quad 1 \quad 0 \quad 1
\end{align*}
\]

\[
\begin{align*}
\text{-2} & \quad -1 \quad 0 \quad 1 \quad 2 \\
\text{-2} & \quad -1 \quad 0 \quad 1 \quad 2
\end{align*}
\]
Projecting LIPs of a single variable

We have $k_1 = 5$ and $d_1 = 3$, so an example of a type-0 and type-1 string is given by

<table>
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<tr>
<th></th>
<th>-2</th>
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<th>0</th>
<th>1</th>
<th>2</th>
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<td>type-0:</td>
<td></td>
<td></td>
<td>1</td>
<td>1</td>
<td>0</td>
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<tr>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
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\[
\{ (x_1 + 2)(x_1 - 0)(x_1 - 1) \}_L \geq 0 \quad -\{ (x_1 + 1)(x_1 - 0)(x_1 - 2) \}_L \geq 0
\]
Projecting LIPs of a single variable

We have $k_1 = 5$ and $d_1 = 3$, so an example of a type-0 and type-1 string is given by

\[
\begin{array}{cccccc}
-2 & -1 & 0 & 1 & 2 & \\
\text{type-0:} & 1 & 0 & 1 & 1 & 0 \\
\end{array}
\quad \quad \quad
\begin{array}{cccccc}
-2 & -1 & 0 & 1 & 2 & \\
\text{type-1:} & 0 & 1 & 1 & 0 & 1 \\
\end{array}
\]

\[
\{(x_1 + 2)(x_1 - 0)(x_1 - 1)\}_L \geq 0 \quad -\{(x_1 + 1)(x_1 - 0)(x_1 - 2)\}_L \geq 0
\]

Type-0 constraint \quad Type-1 constraint
Theorem

Given $x_1$ and any $d_1 \in \{1, \ldots, k_1 - 2\}$, the collection of all type-0 and type-1 constraints of size $d_1$ completely characterizes the projection of the $k_1$ linearized LIPs onto the first $d_1$ dimensions, and define all facets of $DP(d_1, 0)$. 
Projecting LIPs of a single variable

**Theorem**

Given $x_1$ and any $d_1 \in \{1, \ldots, k_1 - 2\}$, the collection of all type-0 and type-1 constraints of size $d_1$ completely characterizes the projection of the $k_1$ linearized LIPs onto the first $d_1$ dimensions, and define all facets of $DP(d_1, 0)$.

**Theorem**

For $DP(d_1, 0)$ with $d_1 \geq 2$, every realization of $x_1$ corresponds to an extreme point.
Projecting LIPs of a single variable

\[ DP(3, 0) \]

\[ DP(2, 0) \]

\[ DP(1, 0) \] is the set \(-2 \leq x_1 \leq 2\). Here the "middle" realizations do not have corresponding extreme points.

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**Discrete polytopes - summary of results, round 3**

\[ DP(d_1, d_2) \text{ for } x_1 \in \{\theta_{11}, \ldots, \theta_{1k_1}\}, \ x_2 \in \{\theta_{21}, \ldots, \theta_{2k_2}\} . \]

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<tr>
<th>(d_1)</th>
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</tr>
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</table>
| \(d_1 = k_1 - 1\) | LIPs | LIPs | LIPs | ????

\[ p(x_1, x_2) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} a_{ij} x_1^i x_2^j \]
Useful LIP property 2

Given $x_1 \in \{\theta_{11}, \ldots, \theta_{1k_1}\}$ and $x_2 \in \{\theta_{21}, \ldots, \theta_{2k_2}\}$, suppose that we multiply each of the $k_1$ LIPs of $x_1$ by each of the $k_2$ LIPs of $x_2$, and set the linearized terms nonnegative.

The set of $k_1 k_2$ linearized, nonnegative LIPs in $k_1 k_2 - 1$ variables describes a polytope having $k_1 k_2$ extreme points, and each extreme point corresponds to some $(\theta_{1i}, \theta_{2j})$ with $w_{ij} = x_1^i x_2^j$ for all $i = 1, \ldots, k_1 - 1$, $j = 1, \ldots, k_2 - 1$. 
Recall the optimization problem

How to solve the below as a linear program?

minimize \( \{(x_1 - 1)^2 + (x_2 - 1)^2 : x_1 \in \{0, 1, 2\}, \ x_2 \in \{0, 1, 2, 3\}\} \)

Desire a polytope having 12 extreme points to correspond to the 12 realizations of \((x_1, x_2)\), and all linearized variables to equal to their intended products at the extreme points.
Recall the optimization problem

How to solve the below as a linear program?

minimize \( \{(x_1 - 1)^2 + (x_2 - 1)^2 : x_1 \in \{0, 1, 2\}, x_2 \in \{0, 1, 2, 3\}\} \)

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\((3 \text{ LIPs for } x_1) \otimes (4 \text{ LIPs for } x_2)\)
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\[
\text{minimize } \{(x_1 - 1)^2 + (x_2 - 1)^2 : x_1 \in \{0, 1, 2\}, \ x_2 \in \{0, 1, 2, 3\}\}
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\[(3 \text{ LIPs for } x_1) \otimes (4 \text{ LIPs for } x_2)\]

But do we really want terms including \(x_2^3\)?
Discrete polytopes - summary of results, round 4

\[ DP(d_1, d_2) \] for \( x_1 \in \{ \theta_{11}, \ldots, \theta_{1k_1} \} \), \( x_2 \in \{ \theta_{21}, \ldots, \theta_{2k_2} \} \).

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\[ p(x_1, x_2) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} a_{ij} x_1^i x_2^j \]
Useful LIP property 3

Given \( x_1 \in S_1 = \{\theta_{11}, \ldots, \theta_{1k_1}\} \) and any polytope \( P \) in variables, say \( y \) not including \( x_1 \), suppose that we multiply each of the \( k_1 \) LIPs of \( x_1 \) by each of the inequalities defining \( P \) and then linearize the resulting terms. Every extreme point of the higher-dimensional space has \( x_1 \) realizing one of its discrete values, and all linearized variables equal to their intended products.
Discrete polytopes - summary of results, round 5

\[ DP(d_1, d_2) \text{ for } x_1 \in \{\theta_{11}, \ldots, \theta_{1k_1}\}, \ x_2 \in \{\theta_{21}, \ldots, \theta_{2k_2}\} \]

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\[
p(x_1, x_2) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} a_{ij} x_1^i x_2^j
\]
Using projections to linearize the product of two variables

We can acquire $DP(k_1 - 1, d_2)$ for $d_2 \in \{2, ..., k_2 - 2\}$ using the polytopes $DP(0, d_2)$ that we just obtained.
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Note: $DP(k_1 - 1, 0)$
Using projections to linearize the product of two variables

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LIPs for $x_1$
Using projections to linearize the product of two variables

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LIPs for $x_1$  Projected LIPs for $x_2$
We can acquire $DP(k_1 - 1, d_2)$ for $d_2 \in \{2, \ldots, k_2 - 2\}$ using the polytopes $DP(0, d_2)$ that we just obtained.

Note: $DP(k_1 - 1, 0) \otimes DP(0, d_2) = DP(k_1 - 1, d_2)$

LIPs for $x_1$  \quad Projected LIPs for $x_2$
Again recall the optimization problem

How to solve the below as a linear program?

minimize \( \{(x_1 - 1)^2 + (x_2 - 1)^2 : x_1 \in \{0, 1, 2\}, \ x_2 \in \{0, 1, 2, 3\}\} \)

Desire a polytope having 12 extreme points to correspond to the 12 realizations of \((x_1, x_2)\), and all linearized variables to equal to their intended products at the extreme points.

\((3 \text{ LIPs for } x_1) \otimes (4 \text{ LIPs for } x_2)\)

But do we really want terms including \(x_2^3\)?

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Overview
Focus on a single variable
Focus on two variables
Question and final comments

Property 2 of LIPs
Property 3 of LIPs
Use of projections

Again recall the optimization problem

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### Overview
Focus on a single variable
Focus on two variables
Question and final comments
Property 2 of LIPs
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### Discrete polytopes - summary of results, round 6

**$DP(d_1, d_2)$ for $x_1 \in \{\theta_{11}, \ldots, \theta_{1k_1}\}$, $x_2 \in \{\theta_{21}, \ldots, \theta_{2k_2}\}$.**

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$$p(x_1, x_2) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} a_{ij} x_1^i x_2^j$$

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Can we characterize the final (??????)?

Multiplying the inequalities of the projected LIPs for $x_1$ by the inequalities of the projected LIPs for $x_2$ gives valid inequalities for $DP(d_1, d_2)$ when $2 \leq d_1 \leq k_1 - 2$ and $2 \leq d_2 \leq k_2 - 2$, but does not fully characterize the set.
Discrete polytopes - summary of results, two comments

$$DP(d_1, d_2)$$ for $$x_1 \in \{\theta_{11}, \ldots, \theta_{1k_1}\}$$, $$x_2 \in \{\theta_{21}, \ldots, \theta_{2k_2}\}$$.

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$$p(x_1, x_2) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} a_{ij} x_1^i x_2^j$$
Two closing comments

- Can characterize products of binary $x_1$ and discrete $x_2$. The simplest case has $d_2 = 2$. The cubic function

$$a_{10} x_1 + a_{01} x_2 + a_{11} x_1 x_2 + a_{02} x_2^2 + a_{12} x_1 x_2^2$$

can be modeled using the linearized function

$$a_{10} x_1 + a_{01} x_2 + a_{11} w_{11} + a_{02} w_{02} + a_{12} w_{12}.$$ 

- As a consequence, can characterize products of continuous $x_1$ and discrete $x_2$. 

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