Spatial Branch-and-Bound for the Alternating Current Optimal Power Flow (ACOPF) Problem

Introduction

ACOPF is an important problem in power systems operations: determine a minimum cost dispatch of electric generation and network assets over a single time period while satisfying demand and engineering constraints. A key modelling obstacle is the incorporation of alternating current, as even with some simplifying assumptions the relationship between relevant variables is fundamentally nonlinear. We adopt a Spatial Branch-and-Bound approach to establish a near optimal global solution, and introduce several innovations to improve performance.

Background

Generically the ACOPF is NP-hard (see [5]), and algorithms used in industry cannot achieve guaranteed convergence to an optimal solution or can establish lower bounds on minimal cost dispatch. As practical problems are of tremendous size, involving tens of thousands of variables, establishing global optimality is a daunting task. The current work (see [1],[5]) involves a previously uncommon formulation in rectangular coordinates, which essentially casts ACOPF as a nonconvex complex QCQP with bounds on all variables. Since the Lagrangian dual associated with any QCQP can be solved via Semidefinite Programming (SDP), lower bounds can be established efficiently. In certain cases where there is no duality gap, primal optimal solutions can be extracted from the dual problem. Researchers have focused on conditions that can guarantee such an occurrence a priori, but no such guarantees can be made for all practical problems. Our work is motivated by instances of ACOPF with substantial duality gap, thus necessitating better lower bounds. In the vein of some related ACOPF literature (see [3,7]), we adopt a spatial branch-and-bound approach. Figure 1 illustrates this generic approach for depth-first search.

Variable Selection

We suggest selecting variable bound partitions that seek to minimize worst-case violation of the relaxed rank constraint in the lower bound problem. We take a local view with this simple observation:

A nonzero Hermitian Positive Semi-definite $\mathbf{n} \times \mathbf{n}$ (n>1) has rank one if all its 2 x 2 principal minors are zero. Therefore we can focus on the principal minor constraint between two buses. Specifically, we find the convex hull of the feasible points:

$$W + jT \geq 0$$

$$\text{rank}(W + jT) = 1$$

$$\text{det}(W + jT) = \text{det}(W + jT) = 0$$

The hull can be obtained by relaxing the rank constraint and adding two linear valid inequalities, for which we have closed-form representation. These inequalities strengthen the Lagrangian Dual-based relaxation and ensure convergence towards a rank-one solution. The convex hull of such a set is shown in Figure 2. The highlighted curve is the boundary of the following constraints:

$$W_{11} W_{22} + W_{12} W_{21} \leq 1.5$$

$$0.5 \leq W_{12} \leq 1.25$$

The two planes are the boundaries of the two linear valid inequalities that must be added to obtain the convex hull.

Formulation

$$\min \sum_{i,j} c_{ij}(W_{i,j} + D_{i,j}) + c_{i}(W_{i,i} + D_{i,i})$$

subject to

$$P + jQ = \text{diag}(V V^*)$$

$$P_{\text{max}} + jQ_{\text{max}} \leq P + j(D + jP) \leq P_{\text{max}} + jQ_{\text{max}}$$

$$\text{Im}(V_{i,j}) \leq \text{Im}(P_{\text{max}}) \text{Re}(V_{i,j})$$

The decision variables are real (P) and reactive (Q) powers, and complex voltage (V) at each bus. All other terms are parameters. * is the complex conjugate operator.

Following the approach of Bai et al. [1], we can represent the constraints in higher dimension using the Hermitian matrix $W$: $W + jT \geq 0$, $\text{rank}(W + jT) = 1$, where $W_{ij} = \text{Re}(V_{i,j})$, $T_{ij} = \text{Im}(V_{i,j})$. We exploit network topology in the positive semi-definite constraint (see [2,6]) and solve a sparse formulation to improve solutions times for lower bounds.

Bound Tightening

We develop some inferences for bound tightening or domain reduction. These are used after each branch decision, and add virtually no additional overhead. One procedure is to extract implied variable bounds on $x$ from the constraints:

$$ax^2 + xy + c \leq 0$$

$$L \leq y \leq U$$

This generic structure can be used to relate power and voltage in various ways. For instance, when $a > 0$, $L \leq |x| \leq U$, $U < 0$ we can infer that:

$$\sqrt{-\frac{a}{2a}} \leq x \leq \sqrt{-\frac{a}{2a}}$$

Note that for power systems considerations (avoiding the crossing of the ‘nose curve’), we can ignore the lesser root in the quadratic on $x$, yielding tighter bounds.

We also develop an inference over graph cycles in the network topology. Since only nodal angle differences are accounted for, we can use the fact that the sum of all such differences in a cycle must be zero. For instance, since we want $\text{arctan}(\frac{\text{Im}(x)}{\text{Re}(x)}) = \theta_n - \theta_1$, this yields:

$$\text{arctan}(\frac{\text{Im}(x)}{\text{Re}(x)}) \geq 0 \rightarrow \text{arctan}(\frac{\text{Im}(x)}{\text{Re}(x)}) \geq 0$$

Experiments

We present in Table 1 a small selection of our experiments, listing performance of various spatial branch-and-bound algorithms on the instances of Gopalakrishnan et al [3]. As a direct application to simple comparison, we show the number of nodes explored in the branch-and-bound process needed to find an optimal solution within 0.1% of optimality. $\text{spbent}$ is our algorithm without bound tightening, $\text{spbbnt}$ includes bound tightening, and $\text{gbb}$ are the results of Gopalakrishnan et al [3].

For brevity we have omitted data on the effects of network topology, as well as conditions producing severe duality gap (>10%). The IEEE standard test cases have sparse network topology, and we show that as a result the 2 x 2 principal minor-based Second-Order Cone Programming relaxation provides a lower bound almost as effective as the Lagrangian dual. We also show that duality gap can arise when applying line limits on real power, current, or apparent power, and with negative generation costs.

Table 1. Nodes explored before establishing near optimality.

<table>
<thead>
<tr>
<th>Problem Size</th>
<th>Algorithm</th>
<th>Nodes Explored</th>
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<tbody>
<tr>
<td>54</td>
<td>spbbnt</td>
<td>31</td>
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<tr>
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<td>spbbnt</td>
<td>30</td>
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<tr>
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<td>57</td>
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<tr>
<td>21</td>
<td>gbb</td>
<td>48</td>
</tr>
<tr>
<td>150</td>
<td>gbb</td>
<td>150</td>
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</tbody>
</table>

Conclusions

We have developed a spatial branch-and-bound algorithm for ACOPF that has enabled us to explore the potential causes of duality gap. Our variable selection technique relies on valid inequalities derived from the convex hull of a rank-constrained complex semidefinite constraint set, which can be generally applied to bounded QCQP. Our bound tightening technique analyzes nonconvex quadratic constraints in the original space, and are a fast closed-form alternative to solving lower bound problems. Due to the tremendous scale of practical problems, establishing global optimality of ACOPF with severe duality gap will require still more innovation. We are currently investigating cuts to improve the root node, and thus reduce the total amount of nodes explored.

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References


