Network design with equilibrium-driven customer demands
Mahdi Hamzeei and James Luedtke
University of Wisconsin-Madison

Given:
- $N$: set of customers
- $M$: set of facilities
- Demand of customer $i \in N$
- Capacity of facility $j \in M$
- Cost of travel from $i$ to $j$
- Cost of opening facility $j$

Key challenge
The allocation of demands of customers to facilities is the solution of an equilibrium constraint.
Proposed approach

- Formulated as a mixed-integer bilevel program
- Proposed a solution method:
  1. Derived a Lagrangian relaxation
  2. Lagrangian dual problem can be decomposed into subproblems, one for each facility
  3. Subproblems can be solved efficiently.
  4. Solution of the relaxation is used to obtain a feasible solution to the original problem
- Gap between lower and upper bound indicates the strength of the solution
New MIP approaches to the floor layout problem

Joey Huchette, Juan Pablo Vielma, Santanu Dey
• Fundamental problem in VLSI…
  ...that we can’t solve at industry scale with MIP
• Central idea: subproblems have same structure

1. Formulations
   a. Get strong formulations for subproblem
   b. Combine for formulation for master problem

2. Lower-bounds
   a. Solve subproblems to optimality ("MIPing")
   b. Generate lower-bounds for master problem
Computational investigation of generalized intersection cuts

Aleksandr M. Kazachkov

Tepper School of Business, Carnegie Mellon University

Based on joint work with Egon Balas, François Margot, and Selvaprabu Nadarajah
How to choose hyperplanes?

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Best strategy for generating cuts?

Table 2: Percentage gap closed by objectives used for cut LP

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When we are making decisions about uncertain future:

- 2-stage Stochastic Program (SP)
  - Some decisions are made in the first stage before realization of uncertainty
  - The first-stage decisions must be feasible for all possible scenarios
  - May result in costly first-stage solution.

- Chance-Constrained Mathematical Program (CCMP)
  - Study the Single-stage (static) problems
  - We can ignore the outcomes of a small fraction of scenarios.
  - Our decisions do not account for the cost incurred by the recourse action

Can we combine these two models?
- Two-stage CCMP : consider costs of recourse actions for feasible scenarios.
- Two-stage CCMP with Recovery (CCMPR) : model the needs to “recover” from a infeasible scenario.
Solution algorithm

- What has been done:
  
  **2-stage SP**
  - Benders decomposition algorithm
  - Adding feasibility cut and optimality cut to the relaxed master problem

  **Single-stage CCMP**
  - A class of valid inequalities has been derived.
  - Can be solved efficiently by a decomposition algorithm

- However, a simple combination of these two approaches does not yield good results for 2-stage CCMPR.

- What we do:
  - Derive a class of strong valid optimality cuts.
  - Propose a Benders-type decomposition algorithm.
Open-source algebraic modeling in Julia

Miles Lubin
MIT Operations Research Center

MIP 2014
State of the art: AMPL, GAMS, Pyomo, CPLEX Concert, CVX, YALMIP, PuLP, GurobiPy, Gurobi C++, CMPL, AIMMS, ZIMPL, ...

In MIP research we want:

- Ease of use (Python/MATLAB over C/C++)
- Access to low-level solver (callbacks)
- Performance
- (Solver independence)

Can we have it all?
A new modeling language for optimization

- Embedded in Julia, a high-performance high-level language for technical computing
- Familiar syntax for users of Python/MATLAB/AMPL
- As fast as hand-written sparse matrix generators
- Solver-independent B&B callbacks for implementing cutting planes (and more)
  - Gurobi, CPLEX, GLPK
- Derivatives and expression trees for MINLP (without AMPL)
- Open source, easy to extend
A machine learning based approximation of strong branching
in the 2014 Mixed Integer Programming Workshop (MIP 2014)

Alejandro Marcos Alvarez    Quentin Louveaux    Louis Wehenkel

University of Liège

July 21, 2014
Motivation

Strong branching generally
- minimizes the number of nodes of the tree
- is intractable in practice

Idea → approximate strong branching with a more tractable function

This is not new!

Innovation of our approach: we use machine learning techniques
What you will see on our poster

1. discover what machine learning is
2. how machine learning can be used to construct a branching heuristic
3. discover a set of features describing fractional variables (main part of the work)
The Power of a Negative Eigenvalue: Aggregation Cuts for Nonlinear Integer Programming

Sina Modaresi
MIT/University of Pittsburgh

Joint Work with
Juan Pablo Vielma (Massachusetts Institute of Technology)

Mixed Integer Programming Workshop, Columbus, July 2014
• “Extended” Aggregation Technique (Yildiran, 2009)

• Characterizing convex hull of a set defined by two quadratic inequalities

\[ S := \{ x \in \mathbb{R}^n : x^T Q_i x + 2b_i^T x + \gamma_i < 0, \quad i = 1, 2 \} \]

• No specific requirement on the two quadratics

• Convex hull characterization is independent of the geometry of the set
Example. Let

\[ S := \{(x, t) \in \mathbb{R}^2 : x^2 - t^2 + 2x + 2 \leq 0, \quad -x^2 + t^2 + 2x - 2 \leq 0\} \]
• Characterizing convex hull of a set defined by a conic and a quadratic inequality

\[ S := \{ x \in \mathbb{R}^n : \| Ax - c \|_2 < a^T x - d \]
\[ x^T Q x + 2b^T x + \gamma < 0 \}, \]

using the same aggregation technique
On Blocking and Anti-Blocking Polyhedra in Infinite Dimensions

Luis Rademacher
Alejandro Toriello
Juan Pablo Vielma
Motivating Question

- For infinite horizon planning:
  Infinite objective function can be avoided by using either
  - a discount $\alpha \in (0,1)$
    \[\sum_{t=1}^{\infty} \alpha^t c_t x_t\]
  - a cumulative average
    \[\lim_{n \to \infty} \sup \frac{1}{n} \sum_{t=1}^{n} c_t x_t\]

- Drawbacks: distant future or near future have low influence, respectively.
Our results

• Use complementary slackness as a notion of optimality with (possibly) infinite objective value: For packing-covering problems (blocking anti-blocking polyhedra):
  – existence of optimal primal-dual pairs
  – equivalent to time-localized optimality
  – integrality of extreme points for natural problems.
MILP model for a real-world hydro-power unit-commitment problem

Pascale Bendotti\textsuperscript{1}, Claudia D’Ambrosio\textsuperscript{2}, Grace Doukopoulos\textsuperscript{1}, Arnaud Lenoir\textsuperscript{1}, Leo Liberti\textsuperscript{2} Youcef Sahraoui\textsuperscript{1,2}

\textsuperscript{1}OSIRIS, EDF R&D
\textsuperscript{2}LIX, Ecole Polytechnique

MIP workshop - Ohio State University
July 21st, 2014
Challenge

Solve a large-scale MILP with real-world data.

- **Goals**
  - convergence to optimality
  - limited resolution time-frame
  - more refined model to fit real description of operations

- **Obstacles**
  - infeasibilities
  - numerical difficulties
  - long resolution times and intractabilities

→ identify and cope with obstacles
Approach, 1st results and future work

1. Approach:
   Simplify model by using notably less binary variables to get first understanding

2. Results for infeasibilities (with Cplex conflict refiner):
   - discard (or modify) data
   - relax or penalize constraints (or consider as 2nd objective)

3. Future work:
   - dealing with numerical instabilities
   - understanding poor resolution performance
   - solving the real problem: bi-objective, decomposition
• Master Knapsack Problem $K(n)$ is the integer solutions to 
  \[ t_1 + 2t_2 + \ldots + nt_n = n, \text{ "the knapsack equation"} \]

• Knapsack facets $\xi t \leq 1$ (assume $\xi_1 = 0$.)

• A knapsack facet is a $1/k$-facet if coefficients $\xi_i \in \{0/k, 1/k, \ldots, k/k\} \cup \{1/2\}$ for all $i = 1, \ldots, n$. 
• Subproblem = general integer knapsack problem

• Every integer knapsack problem is a projection of $K(n)$

$$5t_5 + 6t_6 + 4t_4 \leq 10$$

$$\iff 5t_5 + 6t_6 + 4t_4 + t_1 = 10$$

$$\iff t_1 + 2t_2 + \ldots + 10t_{10} = 10 \quad \text{and} \quad t_2 = t_3 = t_7 = t_8 = t_9 = t_{10} = 0$$

• A knapsack facet $\xi.t \leq 1$ is projected into a valid inequality

$$\xi_{\{4,5,6\}} t_{\{4,5,6\}} \leq 1$$

• Assume $l << n$. 
• Separate > 75% of facets in $O(l^4 = l^{k/2})$

  – Independent of $n$

  – Small $k$ are good enough
Facets of the bound-relaxed two-integer knapsack

Ricardo Fukasawa, Laurent Poirrier, Álinson Xavier

Department of Combinatorics and Optimization
University of Waterloo, Canada

July 2014
We study the facial structure of the polyhedron

\[
P = \text{conv}\left\{(x, s) \in \mathbb{Z} \times \mathbb{R}_+^n : x = f + \sum_{i=1}^{n} r_i s_i, s_1 \in \mathbb{Z}\right\},
\]

where \( n \in \mathbb{Z}_+ \), \( f \in \mathbb{Q} \setminus \mathbb{Z} \) and \( r \in \mathbb{Q}^n \).
We study the facial structure of the polyhedron

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where \( n \in \mathbb{Z}_+, f \in \mathbb{Q} \setminus \mathbb{Z} \) and \( r \in \mathbb{Q}^n \).

- Two-integer knapsack, non-negativity of an integer variable is relaxed.
- Single-row corner relaxation, integrality of a non-basic variable is kept.
We study the facial structure of the polyhedron

\[ P = \text{conv} \left\{ (x, s) \in \mathbb{Z} \times \mathbb{R}^n_+ : x = f + \sum_{i=1}^n r_i s_i, s_1 \in \mathbb{Z} \right\}, \]

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- Can provide cutting planes for general mixed-integer problems.
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- Two-integer knapsack, non-negativity of an integer variable is relaxed.
- Single-row corner relaxation, integrality of a non-basic variable is kept.
- Can provide cutting planes for general mixed-integer problems.

Let \( S = \mathbb{Z} \times \mathbb{Z}_+ \). We can rewrite

\[ P = \left\{ (x, s) \in S \times \mathbb{R}_+^n : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} + \begin{pmatrix} r_1 \\ 1 \end{pmatrix} s_1 + \sum_{i=2}^n \begin{pmatrix} r_i \\ 0 \end{pmatrix} s_i \right\}. \]

- Correspondence between \( S \)-free sets and valid inequalities for \( P \).
Main Results

- All facet-defining inequalities come from:
  - Maximal splits unbounded along the line \( f_0 + \lambda \binom{r_1}{1} \)
  - Maximal \( S \)-free wedges with apex on this same line.
Main Results

- All facet-defining inequalities come from:
  - Maximal splits unbounded along the line \( \left( \frac{f}{0} \right) + \lambda \left( \frac{r_1}{1} \right) \)
  - Maximal \( S \)-free wedges with apex on this same line.

- At most one wedge for each facet of

\[
A = \text{conv}\{x \in \mathbb{Z} \times \mathbb{Z}_+: x_1 - r_1 x_2 \leq f\}, \text{ or } \\
B = \text{conv}\{x \in \mathbb{Z} \times \mathbb{Z}_+: x_1 - r_1 x_2 \geq f\}.
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Main Results

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- Algorithms to:
  - Enumerate the facets of \( A, B \), therefore all facet-defining wedges.
  - Calculate trivial lifting coefficients for additional integer variables.
Main Results

◊ All facet-defining inequalities come from:
  ◇ Maximal splits unbounded along the line \( f_0 + \lambda \left( \begin{array}{c} r_1 \\ 1 \end{array} \right) \)
  ◇ Maximal \( S \)-free wedges with apex on this same line.

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◊ Algorithms to:
  ◇ Enumerate the facets of \( A, B \), therefore all facet-defining wedges.
  ◇ Calculate trivial lifting coefficients for additional integer variables.

◊ Wedge cuts generalize MIR and 2-step MIR cuts.
◊ Upper bound on the split rank.
Main Results

- All facet-defining inequalities come from:
  - Maximal splits unbounded along the line \( \begin{pmatrix} f \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} r_1 \\ 1 \end{pmatrix} \)
  - Maximal \( S \)-free wedges with apex on this same line.

- At most one wedge for each facet of

\[
A = \text{conv}\{ x \in \mathbb{Z} \times \mathbb{Z}_+ : x_1 - r_1 x_2 \leq f \}, \quad \text{or} \\
B = \text{conv}\{ x \in \mathbb{Z} \times \mathbb{Z}_+ : x_1 - r_1 x_2 \geq f \}.
\]

- Algorithms to:
  - Enumerate the facets of \( A, B \), therefore all facet-defining wedges.
  - Calculate trivial lifting coefficients for additional integer variables.

- Wedge cuts generalize MIR and 2-step MIR cuts.
- Upper bound on the split rank.
- Computational experiments.
Generating Multi-row Simplex Cuts on Higher Dimensional Spaces

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Joint work with Endre Boros
Outline

1. Corner polyhedron
2. Maximal lattice-free convex sets
3. Numerical experiment
## Preliminaries

### Corner Polyhedron

Let the corner polyhedron be represented as follows:

\[ C = \text{conv} \left\{ \begin{pmatrix} x \\ s \end{pmatrix} \in \mathbb{Z}^m \times \mathbb{R}^k : x = f + \sum_{i=1}^{k} r_i s_i, \ s \geq 0 \right\} \]  (1)

We want to generate a deepest cut in the sense that it minimizes the sum of its coefficients and separates a fractional vector \( f \) and \( C \).

### Intersection Cut

Let \( S \subset \mathbb{R}^m \) be closed, convex and s.t. \( f \in \text{int}(S) \) and \( \forall x \in \mathbb{Z}^m \Rightarrow x \notin \text{int}(S) \). Then, \( \sum_{j=1}^{k} \Phi_S(r_j)s_j \geq 1 \) is valid for (1) that cuts off basic solution \( f \), where \( \Phi_S(r_j) := \inf\{t > 0 : f + r_j/t \in S\}, \ \forall j \).
Maximal lattice-free convex sets

Theorem (Bell and Scarf (1977))

Let \( \{ x \in \mathbb{Z}^m : Ax \leq b \} = \emptyset \) where \( A \in \mathbb{R}^{\ell \times m} \), then there are \( 2^m \) or less inequalities among \( Ax \leq b \), say \( A'x \leq b' \), such that \( \{ x \in \mathbb{Z}^m : A'x \leq b' \} = \emptyset \).

Theorem (Lovasz (1989))

If \( S \) is maximal-lattice free convex set on \( \mathbb{R}^m \) s.t. \( f \in \text{int}(S) \), then \( S = \{ x \in \mathbb{R}^m : A(x - f) \leq 1 \} \) for some \( A \in \mathbb{R}^{\ell \times m} \), where \( \ell \leq 2^m \) can be assumed.
How to generate lattice-free convex sets

Master Problem

\[
\begin{align*}
\min & \quad \sum_{i=1}^{k} \lambda_i \\
\text{s.t.} & \quad \text{for } i = 1, \ldots, k: \\
& \quad A r_i - \lambda_i 1 \leq 0 \\
& \quad \max_{i=1, \ldots, l} (A(x - f)) \geq 1, \quad \forall x \in X \subset \mathbb{Z}^m \\
& \quad A \in \mathbb{R}^{\ell \times m} \\
& \quad \lambda_i \geq 0, \quad i = 1, \ldots, k
\end{align*}
\]

(2)

Note: Problem (2) is a Fixed-Dimensional Semi-Infinite Mixed-Integer Disjunctive Program for fixed number of simplex rows where \( \ell \leq 2^m \).

Separation Problem

\[
\begin{align*}
\max \quad & \quad \epsilon \\
\text{s.t.} & \quad A^i x + e \epsilon \leq 1 + A^i f \\
& \quad x \in \mathbb{Z}^m \\
& \quad \epsilon \geq 0
\end{align*}
\]

(3)

Note: Here \( \bar{A}^i \) denote the optimal solution of Problem (2) at iteration \( i \) for set \( X \subset \mathbb{Z}^m \).

Let \( \bar{\lambda} \) be an optimal solution to Problem (2) and optimum objective value of the corresponding separation problem is 0. Then, \( \sum_{j=1}^{k} \bar{\lambda}_j s_j \geq 1 \) is a valid inequality for (1).
Diamonds in the rough

Figure: aflow30a

Figure: sp97ar

Figure: protfold

Figure: mzzv11
Mixed-integer conic set: \( S := \{ x \in \mathbb{K} : Ax = b, x_j \in \mathbb{Z} \ \forall j \in J \} \).

Two-term disjunctive relaxation: \( C_1 \cup C_2 \)

where \( C_i := \{ x \in \mathbb{K} : Ax = b, c_i^T x \geq c_{i,0} \} \).

Strong disjunctive cuts for \( S \) through a study of \( \overline{\text{conv}}(C_1 \cup C_2) \)

Main contribution: Explicit and simple description of \( \overline{\text{conv}}(C_1 \cup C_2) \) when \( \mathbb{K} \) is the second-order cone.
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**Strong disjunctive cuts** for \( S \) through a study of \( \text{conv}(C_1 \cup C_2) \)

Main contribution: **Explicit and simple** description of \( \text{conv}(C_1 \cup C_2) \) when \( K \) is the second-order cone.
PART 1: $C_1 \cup C_2$ where $C_i = \{x \in \mathbb{K} : c_i^T x \geq c_{i,0}\}$.

- We derive a family of convex valid inequalities that characterize $\overline{\text{conv}}(C_1 \cup C_2)$:

$$2c_{2,0} - (\beta c_1 + c_2)^T x \leq \sqrt{((\beta c_1 - c_2)^T x)^2 + \mathcal{N}(\beta) \left( x_n^2 - \|\bar{x}\|_2^2 \right)}.$$ 

- We show that these inequalities admit an SOC expression under a suitable “disjointness” assumption:

$$\mathcal{N}(\beta) x + 2(c_2^T x - c_{2,0}) \begin{pmatrix} \beta \bar{c}_1 - \bar{c}_2 \\ -\beta c_{1,n} + c_{2,n} \end{pmatrix} \in \mathbb{K}.$$ 

- We identify cases where a single one of these inequalities is sufficient to describe $\overline{\text{conv}}(C_1 \cup C_2)$.

PART 2: $C_1 \cup C_2$ where $C_i = \{x \in \mathbb{K} : Ax = b, c_i^T x \geq c_{i,0}\}$.

- We identify a set of conditions under which a single inequality of the type above is sufficient to describe $\overline{\text{conv}}(C_1 \cup C_2)$.

- In particular, we can characterize the convex hull of
  - all two-term disjunctions on ellipsoids and paraboloids,
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